

GRADED  $F$ -MODULES AND LOCAL COHOMOLOGY

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ABSTRACT. Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  of characteristic  $p > 0$ , let  $\mathfrak{m} = (x_1, \dots, x_n)$  be the maximal ideal generated by the variables, let  ${}^*E$  be the naturally graded injective hull of  $R/\mathfrak{m}$  and let  ${}^*E(n)$  be  ${}^*E$  degree shifted downward by  $n$ . We introduce the notion of graded  $F$ -modules (as a refinement of the notion of  $F$ -modules) and show that if a graded  $F$ -module  $\mathcal{M}$  has zero-dimensional support, then  $\mathcal{M}$ , as a graded  $R$ -module, is isomorphic to a direct sum of a (possibly infinite) number of copies of  ${}^*E(n)$ .

As a consequence, we show that if the functors  $T_1, \dots, T_s$  and  $T$  are defined by  $T_j = H_{I_j}^{i_j}(-)$  and  $T = T_1 \circ \dots \circ T_s$ , where  $I_1, \dots, I_s$  are homogeneous ideals of  $R$ , then as a naturally graded  $R$ -module, the local cohomology module  $H_{\mathfrak{m}}^{i_0}(T(R))$  is isomorphic to  ${}^*E(n)^c$ , where  $c$  is a finite number. If  $\text{char } k = 0$ , this question is open even for  $s = 1$ .

## 1. INTRODUCTION

Throughout this paper,  $R = k[x_1, \dots, x_n]$  is a polynomial ring over a field  $k$ ,  $\mathfrak{m} = (x_1, \dots, x_n)$  is the maximal homogeneous ideal, and  ${}^*E$  is the  ${}^*$ injective hull of  $R/\mathfrak{m}$ .

It is well-known that  $H_{\mathfrak{m}}^i(H_I^j(R))$ , for an ideal  $I$  of  $R$ , and more generally,  $H_{\mathfrak{m}}^{i_0}(T(R))$  where  $T$  is the composition of functors  $T_1, \dots, T_s$  with  $T_j = H_{I_j}^{i_j}(-)$ , are isomorphic to a direct sum of a finite number of copies of  $E$  (the ungraded injective hull of  $R/\mathfrak{m}$ ). In the case  $\text{char } k > 0$ , this is due to Huneke and Sharp [3], in the case  $\text{char } k = 0$ , this is due to Lyubeznik [4].

If  $I_1, \dots, I_s \subset R$  are homogeneous, the local cohomology module  $H_{\mathfrak{m}}^{i_0}(T(R))$  acquires a natural grading. This paper is motivated by the following question: *How is this grading related to the natural grading on  ${}^*E$ ?* We show in Theorem 3.4 that if  $\text{char } k > 0$ , then  $H_{\mathfrak{m}}^{i_0}(T(R))$  is isomorphic, as a graded  $R$ -module, to a direct sum of a finite number of copies of  ${}^*E(n)$ , that is  ${}^*E$  degree shifted downward by  $n$ . If  $\text{char } k = 0$ , the question is open even for  $s = 1$ .

Our proof is based on a new notion of graded  $F$ -modules which is a fairly straightforward graded version of  $F$ -modules introduced in [5]. The local cohomology modules  $H_I^j(R)$  and  $H_{\mathfrak{m}}^{i_0}(T(R))$  carry a natural structure of graded  $F$ -modules. Our main result (Theorem 3.3) says that a graded  $F$ -module supported in dimension 0 is isomorphic, as a graded  $R$ -module, to a direct sum of copies of  ${}^*E(n)$ . The above-mentioned Theorem 3.4 about  $H_{\mathfrak{m}}^{i_0}(T(R))$  is a straightforward consequence of this result.

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## 2. PRELIMINARIES

For the rest of this paper, we assume that  $k$  is of characteristic  $p > 0$ . The Frobenius homomorphism  $R \xrightarrow{r \mapsto r^p} R'$ , where  $R'$  is another copy of  $R$ , induces the Frobenius functor  $F : R\text{-mod} \rightarrow R\text{-mod}$  as the pull back functor, that is  $F(M) = R' \otimes_R M$  and  $F(M \xrightarrow{f} N) = (R' \otimes_R M \xrightarrow{\text{id} \otimes_R f} R' \otimes_R N)$ . We follow [5] for the  $F$ -module theory.

**Definition 2.1.** [5, Definition 1.1] *An  $F$ -module is an  $R$ -module  $\mathcal{M}$  equipped with an  $R$ -module isomorphism  $\theta : \mathcal{M} \rightarrow F(\mathcal{M})$  called the structure morphism of  $\mathcal{M}$ .*

*A homomorphism of  $F$ -modules is an  $R$ -module homomorphism  $f : \mathcal{M} \rightarrow \mathcal{M}'$  such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{M}' \\ \theta \downarrow & & \downarrow \theta' \\ F(\mathcal{M}) & \xrightarrow{F(f)} & F(\mathcal{M}'), \end{array}$$

where  $\theta$  and  $\theta'$  are the structure morphisms of  $\mathcal{M}$  and  $\mathcal{M}'$ .

Observe that the ring  $R = k[x_1, \dots, x_n]$  has a natural grading  $R = \bigoplus_{i \in \mathbb{N}} R_i$  (as a  $\mathbb{Z}$ -module) such that  $R_i$  consists of all homogeneous polynomials in  $x_1, \dots, x_n$  of degree  $i$ . Recall that a graded  $R$ -module is an  $R$ -module  $M$  together with a decomposition  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  (as a  $\mathbb{Z}$ -module) such that  $R_i M_j = M_{i+j}$  for all  $i, j \in \mathbb{Z}$ . Recall if  $M$  and  $N$  are both graded  $R$ -modules, then a homomorphism  $\varphi : M \rightarrow N$  is degree preserving if  $\varphi(M_i) \subseteq N_i$  for all  $i \in \mathbb{Z}$ .

If  $\mathcal{M}$  is graded, we define the grading of  $F(\mathcal{M})$  by  $\deg r \otimes x = \deg r + p \cdot \deg x$  for all homogeneous  $r \in R$  and  $x \in \mathcal{M}$ . Now we introduce a definition of graded  $F$ -modules as follows:

**Definition 2.2.** *An  $F$ -module  $(\mathcal{M}, \theta)$  is graded if  $\mathcal{M}$  is a graded  $R$ -module and the structure isomorphism  $\theta : \mathcal{M} \rightarrow F(\mathcal{M})$  is degree preserving. A homomorphism of graded  $F$ -modules  $f : \mathcal{M} \rightarrow \mathcal{M}'$  is a degree preserving  $F$ -module homomorphism.*

**Example 2.3.** *The canonical  $F$ -module structure on  $R$  defined by the  $R$ -module isomorphism  $\theta : R \xrightarrow{r \mapsto r \otimes 1} F(R)$  [5, Page 72] makes  $(R, \theta)$  a graded  $F$ -module.*

The theory of  $F$ -modules developed in [5] can be developed in this graded version without difficulty. In particular, it is easily seen that the category of graded  $F$ -modules is abelian. The facts we need in this paper are the following with the use of the standard terminology in [2, Section 3.6]:

**Theorem 2.4.** *If  $\mathcal{M}$  is a graded  $F$ -module, then there is an induced graded  $F$ -module structure on the local cohomology modules  $H_I^i(\mathcal{M})$  for any homogeneous ideal  $I$  of  $R$ .*

*Proof.* Since the ordinary local cohomology can be computed using  $*$ injective resolutions [1, Corollary 12.3.3], the proof is basically the same as in [5, Example 1.2(b)] except that instead of injective resolutions one uses  $*$ injective ones.  $\square$

**Theorem 2.5.** *If  $\mathcal{M}$  is a graded  $F$ -module such that  $\dim_R \text{Supp } \mathcal{M} = 0$ , then  $\mathcal{M}$  is a  $*$ injective  $R$ -module.*

*Proof.* A proof of this is, with minor and straightforward modifications, the same as the proof of the  $\dim_R \text{Supp } \mathcal{M} = 0$  case of [5, Theorem 1.4]. Modifications involve choosing the elements  $e_i$  and  $e_{i,j}$  homogeneous and in the last step showing that  $M_i$  is isomorphic to  ${}^*E(R/\mathfrak{m})(t)$  where  $t = \deg e_i$  (rather than just  $E(R/\mathfrak{m})$ , as in [5, Theorem 1.4]).  $\square$

### 3. THE MAIN RESULT

From this section on, we will adopt the notation  $F^*$  to represent the Frobenius functor  $F$ . Let us first recall a result about the adjointness between  $F_*^l$  and  $F^{*l}$ . We denote the source and target of  $F^l$  by  $R_s$  and  $R_t$  respectively, that is  $F^l : R_s \rightarrow R_t$ . There are two associated functors

$$F^{*l} : R_s\text{-mod} \rightarrow R_t\text{-mod}$$

such that  $F^{*l}(-) = R_t \otimes_{R_s} -$ , and

$$F_*^l : R_t\text{-mod} \rightarrow R_s\text{-mod}$$

which is the restriction of scalars.

Denote the multi-index  $(i_1, \dots, i_n)$  by  $\bar{i}$ , especially  $\overline{p^l - 1} = (p^l - 1, \dots, p^l - 1)$ . When  $k$  is perfect,  $R_t$  is a free  $R_s$ -module on the  $p^{ln}$  monomials  $e_{\bar{i}} = x_1^{i_1} \cdots x_n^{i_n}$  where  $0 \leq i_j < p^l$  for every  $j$ . Suppose  $M$  is an  $R_t$ -module and  $N$  is an  $R_s$ -module. For each  $f \in \text{Hom}_{R_t}(M, F^{*l}(N))$ , define  $f_{\bar{i}} = p_{\bar{i}} \circ f : F_*^l(M) \rightarrow N$ , where

$$F^{*l}(N) = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N) \xrightarrow{y \mapsto e_{\bar{i}} \otimes_{R_s} p_{\bar{i}}(y)} e_{\bar{i}} \otimes_{R_s} N$$

is the natural projection to the  $\bar{i}$ -component. The duality theorem in [6] says:

**Theorem 3.1.** (Theorem 3.3 in [6]) *When  $k$  is perfect, for every  $R_t$ -module  $M$  and every  $R_s$ -module  $N$ , there is an  $R_t$ -linear isomorphism*

$$\begin{aligned} \text{Hom}_{R_s}(F_*^l(M), N) &\cong \text{Hom}_{R_t}(M, F^{*l}(N)) \\ g_{\overline{p^l - 1}}(-) &\leftarrow (g = \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g_{\bar{i}}(-))) \\ g &\mapsto \oplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} g(e_{\overline{p^l - 1} - \bar{i}}(-))). \end{aligned}$$

We use this duality theorem to prove the following striking result.

**Theorem 3.2.** *Let  $M$  be a graded  $R$ -module. Assume*

- (1)  $\{d \in \mathbb{Z} \mid M_d \neq 0\}$  is finite;
- (2)  $M_{-n} = 0$ .

*Then there is  $s \in \mathbb{N}$  (that depends only on the set  $\{d \in \mathbb{Z} \mid M_d \neq 0\}$ ) such that for any  $l \geq s$  and for any graded  $R$ -module  $N$ , the only degree preserving  $R$ -module map  $f : M \rightarrow F^{*l}(N)$  is the zero map.*

*Proof.* Let  $K$  be the perfect closure of  $k$ . Viewing  $K \otimes_k R$ ,  $K \otimes_k M$  and  $K \otimes_k N$  as the new  $R$ ,  $M$  and  $N$ , we may assume that  $k$  is perfect. Therefore Theorem 3.1 applies and it is sufficient to study the  $\overline{p^l - 1}$  component of the image of  $M$ . Recall that the Frobenius functor  $F^{*l}$  multiplies the grading by  $p^l$ , i.e.

$$\deg r \otimes x = \deg r + p^l \cdot \deg x, \quad (r \in R_t, x \in N \text{ and } r \otimes x \in F^{*l}(N)).$$

Since  $F^{*l}(N) = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N)$  and  $\deg e_{\bar{i}} = \sum_j i_j$ , for every  $d \in \mathbb{Z}$  we have

$$F^{*l}(N)_d = \bigoplus_{\bar{i}} (e_{\bar{i}} \otimes_{R_s} N)_d = \bigoplus_{\bar{i}} e_{\bar{i}} \otimes_{R_s} N_{(d - \deg e_{\bar{i}})/p^l},$$

where the direct sum is taken over those  $\bar{i}$  for which  $(d - \deg e_{\bar{i}})/p^l$  is an integer. Clearly,  $\deg e_{\overline{p^l-1}} = n(p^l - 1)$ . When  $d \neq -n$  and  $l$  is sufficiently large, the fraction  $(d - n(p^l - 1))/p^l$  is not an integer. Hence the coefficient of  $e_{\overline{p^l-1}}$  in  $F^{*l}(N)_d$  is 0. Let  $d$  run through the finite set  $\{d \in \mathbb{Z} \mid M_d \neq 0\}$  and enlarge  $l$  correspondingly, we see that the  $\overline{p^l-1}$  component of the image of  $M$  is 0. Moreover, it is obvious that the selection of  $l$  depends only on the set  $\{d \in \mathbb{Z} \mid M_d \neq 0\}$ . Now Theorem 3.1 induces the conclusion.  $\square$

**Theorem 3.3.** *Let  $\mathcal{M}$  be a graded  $F$ -module supported on  $\mathfrak{m} = (x_1, \dots, x_n)$ . Then  $\mathcal{M}$  as a graded  $R$ -module is a direct sum of a (possibly infinite) number of copies of  ${}^*E(n)$ .*

*Proof.* Since  $\mathcal{M}$  is supported on  $\mathfrak{m}$ , it is  ${}^*$ injective by Theorem 2.5. By [2, Theorem 3.6.3], every  ${}^*$ injective module can be decomposed into a direct sum of modules  ${}^*E(R/\mathfrak{p})(i)$  for graded prime ideals  $\mathfrak{p} \in \text{Spec} R$  and integers  $i \in \mathbb{Z}$ . Since  $\mathcal{M}$  is supported on  $\mathfrak{m}$ , the only  $\mathfrak{p}$  that appears in the decomposition is  $\mathfrak{p} = \mathfrak{m}$ , i.e.  $\mathcal{M} = \bigoplus_i {}^*E(i)^{\alpha(i)}$  where  ${}^*E = {}^*E(R/\mathfrak{m})$  and  $\alpha(i)$  is the (possibly infinite) number of copies of  ${}^*E(i)$ . Let  $\theta : \mathcal{M} \rightarrow F^*(\mathcal{M})$  be the structure isomorphism of  $\mathcal{M}$ . Fix  $i \neq n$ , assume  $\alpha(i) \neq 0$ , i.e.  $\text{soc} {}^*E(i)^{\alpha(i)} \neq 0$ , and apply Theorem 3.2 to  $M = \text{soc} {}^*E(i)^{\alpha(i)}$  and  $N = \mathcal{M}$ . Since the degree of the socle of  ${}^*E(i)$  is  $-i \neq -n$ , we see that the composition of isomorphisms  $F^{*l}(\theta) \circ F^{*(l-1)}(\theta) \circ \dots \circ \theta : \mathcal{M} \rightarrow F^{*l}(\mathcal{M})$  vanishes on  $M$ , i.e.  $\theta$  is not an isomorphism. That is a contradiction. Hence  $\alpha(i) = 0$  when  $i \neq n$ .  $\square$

**Theorem 3.4.** *Let the functors  $T_1, \dots, T_s$  and  $T$  be defined by  $T_j = H_{I_j}^{i_j}(-)$  and  $T = T_1 \circ \dots \circ T_s$ , where  $I_1, \dots, I_s$  are homogeneous ideals of  $R$ . Then as a graded  $R$ -module,  $H_{\mathfrak{m}}^{i_0}(T(R))$  is isomorphic to  ${}^*E(n)^c$  for some  $c < \infty$ .*

*Proof.* The graded  $F$ -module structure on  $R$  in Example 2.3 induces a graded  $F$ -module structure on  $H_{\mathfrak{m}}^{i_0}(T(R))$  by induction on  $s$  via Theorem 2.4. Now Theorem 3.3 gives the desired result.  $\square$

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